CALCULATION OF PLANE EQUILIBRIUM
SHAPES OF THIN RODS BY THE METHOD
OF SELF-EQUILIBRATING VARIANCES

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Along with analytical methods [1], the plane equilibrium states of thin elastic rods are also calculated with numerical methods based on reducing nonlinear boundary problems to Cauchy problems [2] or to a sequence of linear boundary problems [1, 3].

Sequential approximations have been constructed [3] from equations whose nonlinear part is taken equal to its value in the previous approximation. A range of loads has been given for which the sequence of approximations converges.

Popov [1] constructed sequential approximations by linearizing the equilibrium equations for a rod and solving the linearized difference equations directly. The linearized difference equations cannot satisfy the stability conditions for calculating them by "ordinary" trial-and-error methods [4]. Popov [1] used non-monotonic steps [5]. If the linearized difference equations are solved by direct methods, the resultant sequence of approximations might not converge — for example if the same load can correspond to several states, so that the approximations oscillate between them. This deficiency can be compensated only by selecting appropriate initial approximations.

The method of local variations [6] can be used to compute the equilibrium shape of thin rods. The sequences of approximations obtained from this method converge for any load; however convergence can be very slow and can require significant computational resources.

The algorithm we present below is designed to solve for the plane equilibrium shapes of thin elastic rods for "dead" loads. It can solve for equilibrium states of "following" loads only in the case where the "following" loads can be obtained as the limit of a sequence of solution for "dead" loads.

This algorithm computes sequential approximations with monotonically decreasing values of the potential energy. Therefore only those equilibrium states are obtained in which the extremum potential energy from bending and external forces is not a maximum.

The algorithm utilizes a variational-difference formulation of the problem. Sequences of approximations are constructed by minimizing a quadratic functional that approximate the potential energy from bending and external forces. These minimization variants can be interpreted as the construction of sequences of approximate solutions to linearized difference equations. In these sequences, sums of the variances and sums of the moments of the variances of the linearized equations are equal to zero in a sequentially expanding region. Hence the name — the method of self-extinguishing variances.
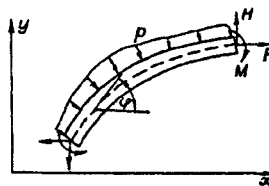

Fig. 1

If the magnitudes of the variations in the target functions are small enough, the decrease in the quadratic functional that approximates the potential energy of bending and external forces also tends to decrease the potential energy. This allows sequences of approximations to be constructed whose convergence is controlled by the fact that they correspond to monotonically decreasing energy sequences. Numerical experiments show that the conditions for calculating the approximations with monotonically decreasing potential energies are established after several iterations, even without introducing any limitations on the magnitude of the variations of the target functions.

1. PROBLEM FORMULATION

The equation for the equilibrium plane states of thin rods [1] are written in the form

$$\frac{dH}{ds} = p \cos \varphi, \ \frac{dF}{ds} = -p \sin \varphi; \tag{1.1}$$

$$\frac{dM}{ds} = H \cos \varphi - F \sin \varphi, \ M = -D\left(\frac{d\varphi}{ds} - \frac{d\varphi_0}{ds}\right); \tag{1.2}$$

$$\frac{dx}{ds} = \cos \varphi, \ \frac{dy}{ds} = \sin \varphi, \tag{1.3}$$

where p is the surface load; F and H are the components of the load vector; M is the bending moment; $D = D(s)$ is the bending rigidity of the rod; and $\varphi(s)$ and $\varphi_0(s)$ are the inclination angles of the rod in the deformed and undeformed states (Fig. 1).

We can set

$$\varphi(0) = x(0) = y(0) = 0. \tag{1.4}$$

without loss of generality. For boundary conditions on φ and s we set

$$[\alpha M + (1 - \alpha)\varphi \,]_{s=l} = \gamma, \tag{1.5}$$

where α equals zero or unity and γ is a specified quantity.

We limit ourselves to the class of problems where the magnitudes of H and F are specified at $s = l$; consequently H(s) and F(s) can be found by integrating Eq. (1.1). We call the load "dead" if H and F do not depend on $\varphi(s)$ and "following" if they do. For "dead" loads, the problem solution reduces to iterating Eqs. (1.2) for boundary conditions (1.4) and (1.5). The coordinates x and y of the rod axis are found by integrating Eqs. (1.3) for conditions (1.4).

The solution of the problem for "dead" loads corresponds to the extremum of the functional

$$\Psi = \frac{1}{2} \int_0^l [D\left(\frac{d(\varphi - \varphi_0)}{ds}\right)^2 - 2(F \cos \varphi + H \sin \varphi)\,]ds + \alpha\gamma\varphi(l) \tag{1.6}$$

in the class of functions φ which satisfy Eqs. (1.4) and (1.5) if $\alpha = 0$. The value of Ψ is the potential energy from bending and external forces.

We divide the rod axis into N intervals with the nodes $s_i = (i - 1)h$ ($i = 1,..., N + 1$, $h = l/N$) and replace the functional (1.6) with the finite-difference form

$$\Psi' = \frac{h}{2} \sum_{i=1}^{N} \left[D_{i+1/2}\left(\frac{\varphi_{i+1} - \varphi_i - \varphi_{i+1}^0 + \varphi_i^0}{h}\right)^2 - 2T_{i+1/2}\right] + \alpha\gamma\varphi_{N+1}, \tag{1.7}$$

where

$$T_{i+1/2} = F_{i+1/2}\cos \varphi_{i+1/2} + H_{i+1/2}\sin \varphi_{i+1/2},$$

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$$\varphi_{i+1/2} = \frac{1}{2}(\varphi_i + \varphi_{i+1}), \quad F_{i+1/2} = \frac{1}{2}(F_i + F_{i+1}), \quad H_{i+1/2} = \frac{1}{2}(H_i + H_{i+1}), \tag{1.8}$$

$$\varphi_1 = 0, \varphi_{N+1} = \gamma, \quad \text{if} \quad \alpha = 0.$$

Below we examine the problem of constructing sequences of approximations with monotonically decreasing magnitudes of the functional (1.7).

2. ELEMENTS OF METHOD TO CONSTRUCT SEQUENCES OF APPROXIMATIONS

Let $\tilde{\varphi}_i (i = 1, 2, ..., N + 1)$ satisfy the conditions (1.8). By replacing φ_i in Eq. (1.7) by $\tilde{\varphi}_i + u_i$ and by assuming that

$$\sin \frac{u_i + u_{i+1}}{2} \approx \frac{u_i + u_{i+1}}{2},$$

$$\cos \frac{u_i + u_{i+1}}{2} \approx 1 - \frac{1}{2} \left(\frac{u_i + u_{i+1}}{2} \right)^2, \tag{2.1}$$

we obtain a quadratic functional relative to u_i

$$\Psi'' = \Psi'(\tilde{\varphi}_i) + \frac{h}{2} \sum_{i=1}^{N} \left[D_{i+1/2} \left(\frac{u_i - u_{i+1}}{2} \right)^2 + \frac{1}{4} \tilde{T}_{i+1/2}(u_i + u_{i+1})^2 - \right.$$

$$\left. \tilde{Q}_{i+1/2}(u_i + u_{i+1}) + 2D_{i+1/2} \frac{\tilde{\varphi}_{i+1} - \tilde{\varphi}_i - \varphi_{i+1}^0 + \varphi_i^0}{h} (u_{i+1} - u_i) \right] + \alpha\gamma u_{N+1}, \tag{2.2}$$

where

$$\tilde{T}_{i+1/2} = F_{i+1/2}\cos \tilde{\varphi}_{i+1/2} + H_{i+1/2}\sin \tilde{\varphi}_{i+1/2}, \quad \tilde{Q}_{i+1/2} = H_{i+1/2}\cos \tilde{\varphi}_{i+1/2} - F_{i+1/2}\sin \tilde{\varphi}_{i+1/2},$$

$$u_1 = 0, u_{N+1} = 0, \quad \text{if} \quad \alpha = 0. \tag{2.3}$$

Under the conditions (2.3), the extremum of the functional (2.2) corresponds to the solution to the system of equations

$$a_i u_{i-1} + 2b_i u_i + c_i u_{i+1} = f_i, i = 2, 3, ..., N,$$
$$u_1 = 0, u_{N+1} = 0, \quad \text{if} \quad \alpha = 0,$$
$$d_N u_N + d_{N+1} u_{N+1} + f_{N+1} = 0, \quad \text{if} \quad \alpha = 1. \tag{2.4}$$

Here

$$a_i = -\frac{1}{h} D_{i-1/2} + \frac{h}{4} \tilde{T}_{i-1/2}, \quad c_i = -\frac{1}{h} D_{i+1/2} + \frac{h}{4} \tilde{T}_{i+1/2},$$

$$2b_i = -\frac{1}{h} (D_{i-1/2} + D_{i+1/2}) + \frac{h}{4} (\tilde{T}_{i-1/2} + \tilde{T}_{i+1/2}), \tag{2.5}$$

$$f_i = -\frac{h}{2} (\tilde{Q}_{i-1/2} + \tilde{Q}_{i+1/2}) + \frac{1}{h} [D_{i-1/2}(\tilde{\varphi}_i - \tilde{\varphi}_{i-1} - \varphi_i^0 + \varphi_{i-1}^0) -$$

$$D_{i+1/2}(\tilde{\varphi}_{i+1} - \tilde{\varphi}_i - \varphi_{i+1}^0 + \varphi_i^0)], i = 2, 3, ..., N;$$

$$d_N = -\frac{1}{h} D_{N+1/2} + \frac{h}{4} \tilde{T}_{N+1/2}, \quad d_{N+1} = \frac{1}{h} D_{N+1/2} + \frac{h}{4} \tilde{T}_{N+1/2},$$

$$f_{N+1} = -\frac{h}{2} \tilde{Q}_{N+1/2} + \frac{1}{h} D_{N+1}(\tilde{\varphi}_{N+1} - \tilde{\varphi}_N - \varphi_{N+1}^0 + \varphi_N^0) + \gamma. \tag{2.6}$$

The quantities f_i ($i = 2, ..., N + 1$) are variances in the extremum conditions of the functional (1.7) when φ_i is replaced by $\tilde{\varphi}_i$ in the functional. The process of calculating solutions to Eqs. (2.4)-(2.6) by "ordinary" trial-and-error methods [4] can be unstable. Analogous difference equations (2.4-2.6) [1] have been solved by nonmonotonic trial-and-error methods [5].

In our methods, sequences of approximations, which minimize the functional (1.7) are constructed from sequences of approximate solutions $\bar{u}_i$ (i = 1, 2, ..., N + 1) to Eqs. (2.4), which satisfy the following conditions:

$$\Psi''(\bar{\varphi}_i) \geq \Psi''(\bar{\varphi}_i + \bar{u}_i), \tag{2.7}$$

where equality is possible only if

$$\bar{u}_i = 0, \; i = 2, ..., N + 1;$$

the equalities $\bar{u}_i = 0$ (i = 2, ..., N + 1) are possible only when $f_i = 0$ (i = 2, ..., N) and $f_{N+1} = 0$ if $\alpha = 1$; i.e., in the case when 1) $\bar{\varphi}_i$(i = 2, ..., N + 1) satisfies the extremum conditions for the functional (1.7); 2) the number of operations to calculate $\bar{u}_i$ (i = 1, 2,..., N + 1), which belong to one grid node does not depend on the number of grid nodes; and 3)

$$|\bar{u}_i| \leq A, \; i = 1, 2, ..., N + 1 \tag{2.8}$$

where A is a specified number.

We note that $\Psi''(\bar{\varphi}_i + \bar{u}_i)$. However, the difference in the quantities $\Psi''(\bar{\varphi}_i + \bar{u}_i)$ and $\Psi'(\bar{\varphi}_i + \bar{u}_i)$ are determined by the error (2.1) in the approximation of the sines and cosines. Therefore, by specifying A in (2.8) small enough, this difference can be made so small that the inequality

$$\Psi'(\bar{\varphi}_i) \geq \Psi'(\bar{\varphi}_i + \bar{u}_i).$$

follows from (2.7). This allows our algorithm to construct sequences of approximations with monotonically decreasing values of the functional (1.7).

Numerical calculations show that conditions for calculating approximations with monotonically decreasing values of the functional (1.7) are established after several approximations, even without introducing the limitation (2.8). Therefore this limitation can be introduced only to accelerate convergence of the sequence of approximations. Here it should be kept in mind that introducing the limitation (2.8) can also slow convergence.

Only one equilibrium state can be found for any specified initial approximation with a given value of A in (2.8). Different equilibrium states, which correspond to the same loads, can be found by considering different initial approximations. In some case different equilibrium states are obtained for the same initial approximations, but with different values of A in (2.8) or if the rod is divided into a different number of elements.

3. APPROXIMATION ALGORITHM WITH A SPECIFIED
ANGLE OF ROTATION AT THE END OF THE ROD

If an angle of rotation is specified at the end of the rods s = l, then $u_1 = u_{N+1} = 0$ in Eqs. (2.4). Therefore the iterative calculation reduces to calculating u_i, i = 2, 3, ..., N.

Numerical calculations show that the approximations usually converge faster, if two types of iterations are alternated. One of them is based on constructing approximate solutions to Eq. (2.4) in the form

$$u_i^0 = 0, \; i = 1, 2, ..., N + 1, \; u_1^{(k)} = \dot{u}_1^{(k)} = u_{N+1}^{(k)} = \dot{u}_{N+1}^{(k)} = 0,$$
$$u_i^{(k)} = u_i^{(k-1)} + \dot{u}_i^{(k)}, \; \dot{u}_i^{(k)} = \gamma^{(k)} + \beta^{(k)} s_i, \; i = 2, ..., k + 2, \tag{3.1}$$
$$u_i^{(k)} = \dot{u}_i^{(k)} = 0, \; i = k + 3, ..., N + 1, \; k = 1, 2, ..., N - 2,$$

where $\gamma^{(k)}$ and $\beta^{(k)}$ are determined sequentially by the conditions for the extremum of the functional (2.2) after u_i in the functional is replaced by $u_i^{(k)}$ for k = 1, 2, ..., N−2. These conditions can be written as variation equations

$$\sum_{i=2}^{k+2} z_i^{(k)} \delta \dot{u}_i^{(k)} = 0, \; z_i^{(k)} = a_i u_{i-1}^{(k)} + 2b_i u_i^{(k)} + c_i u_{i+1}^{(k)} - f_i, \tag{3.2}$$
$$k = 1, 2, ..., N - 2.$$

From Eqs. (3.1) and (3.2) it follows that

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$$\sum_{i=2}^{k+2} z_i^{(k)} = 0, \quad \sum_{i=2}^{k+2} z_i^{(k)} s_i = 0, \quad k = 1, 2, \ldots, N - 2. \tag{3.3}$$

Equations (3.3) mean that the sums of the variances of Eqs. (2.4) and their moments are zero; therefore the sequences (3.1) of the solutions to Eqs. (2.4) correspond to a sequence of variances of these equation with zero sums of the variations and moments in a sequentially expanding region.

Obviously, Eqs. (3.3) can be written in the form

$$z_2^{(1)} = 0, \quad z_3^{(1)} = 0, \quad \sum_{i=2}^{k+1} (z_i^{(k)} - z_i^{(k-1)}) + z_{k+2}^{(k)} = 0,$$

$$\sum_{i=2}^{k+1} (z_i^{(k)} - z_i^{(k-1)}) s_i + z_{k+2}^{(k)} s_{k+2} = 0, \quad k = 2, \ldots, N - 2. \tag{3.4}$$

From Eqs. (3.4) it is easy to find that the coefficients of $\gamma^{(k)}$ and $\beta^{(k)}$ in the equations

$$E_1^{(k)} \gamma^{(k)} + L_1^{(k)} \beta^{(k)} + \chi_1^{(k)} = 0,$$
$$E_2^{(k)} \gamma^{(k)} + L_2^{(k)} \beta^{(k)} + \chi_2^{(k)} = 0, \quad k = 1, 2, \ldots, N - 2 \tag{3.5}$$

are related to the coefficients of $\gamma^{(k-1)}$ and $\beta^{(k-1)}$ by the recurrence formulas:

$$E_1^{(1)} = c_2 + a_3 + 2(b_2 + b_3), \quad L_1^{(1)} = (2b_2 + a_3)s_2 + (2b_3 + c_2)s_3,$$
$$E_2^{(1)} = (2b_2 + c_2)s_2 + (2b_3 + a_3)s_3,$$
$$L_2^{(1)} = (2b_2 s_2 + a_3 s_3)s_2 + (c_2 s_2 + 2b_3 s_3)s_3,$$
$$\chi_1^{(1)} = -(f_2 + f_3), \quad \chi_2^{(1)} = -(f_2 s_2 + f_3 s_3),$$
$$E_1^{(k)} = E_1^{(k-1)} + c_{k+1} + a_{k+2} + 2b_{k+2}, \tag{3.6}$$
$$L_1^{(k)} = L_1^{(k-1)} + a_{k+2}s_{k+1} + (c_{k+1} + 2b_{k+2})s_{k+2},$$
$$E_2^{(k)} = E_2^{(k-1)} + c_{k+1}s_{k+1} + (2b_{k+2} + a_{k+2})s_{k+2},$$
$$L_2^{(k)} = L_2^{(k-1)} + a_{k+2}s_{k+1}s_{k+2} + (c_{k+1}s_{k+1} + 2b_{k+2}s_{k+2})s_{k+2},$$
$$\chi_1^{(k)} = a_{k+2}(\gamma^{(k-1)} + \beta^{(k-1)}s_{k+1}) - f_{k+2}, \quad \chi_2^{(k)} = \chi_1^{(k)} s_{k+2}.$$

From the dependence of $\dot{u}_i^{(k)}$ on s_i in Eq. (3.1) we find

$$h \sum_{i=1}^{N} \left[D_{i+1/2} \left(\frac{\dot{u}_{i+1}^{(k)} - \dot{u}_i^{(k)}}{h} \right)^2 + \frac{1}{4} \tilde{T}_{i+1/2} (\dot{u}_i^{(k)} + \dot{u}_{i+1}^{(k)})^2 \right] \geq \frac{1}{h} (D_* - T_* lh)(\dot{u}_*^{(k)})^2, \tag{3.7}$$

where

$$D_* = \min_i D_{i+1/2}; \quad T_* = \max_i (|F_{i+1/2}| + |H_{i+1/2}|);$$
$$\dot{u}_*^{(k)} = \max_i |\dot{u}_i^{(k)}| \quad (i = 2, \ldots, N).$$

According to Eq. (3.7), if

$$h \leq \frac{D_*}{T_* l} \tag{3.8}$$

and if u_i is replaced by $u_i^{(k)}$ in Eq. (2.2), then the functional (2.2) will be a positive definite quadratic from relative to $\gamma^{(k)}$ and $\beta^{(k)}$. Therefore $\gamma^{(k)}$ and $\beta^{(k)}$ ($k = 1, 2, \ldots, N - 2$) are calculated sequentially according to Eqs. (3.5) and (3.6), and the equations

$$u_i^{(k)} = 0 \quad (i = 1, 2, \ldots, N, \ k = 1, 2, \ldots, N - 2)$$

are possible only if $f_i = 0$ ($i = 1, 2, \ldots, N$), i.e., when $\tilde{\varphi}_i$ ($i = 1, 2, \ldots, N$) corresponds to an extremum of the functional (1.7). Otherwise the sequences of approximations (3.1) correspond to a monotonically decreasing sequence of values of the functional (2.2):

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TABLE 1

N	Shape		
	4	5	6
	$\Psi'(x;y)$		
40	$-6,2447(0,1293; -0,2494)$	$-7,1400(0,0821; -0,5995)$	$51,7565(-0,5483; -0,5641)$
80	$-6,2667(0,1303; -0,2499)$	$-7,1536(0,0827; -0,5917)$	$51,7402(-0,5484; -0,5642)$
120	$-6,2708(0.1304; -0.2501)$	$-7,1565(0.0825; -0.5919)$	$51.7371(-0.5496; -0.5642)$

$$\Psi'(\tilde{\varphi}_i) = \Psi''(\tilde{\varphi}_i) \geqslant \Psi''(\tilde{\varphi}_i + u_i^{(1)}) > \Psi''(\tilde{\varphi}_i + u_i^{(2)}) > ... > \Psi''(\tilde{\varphi}_i + u_i^{(N-2)}). \qquad (3.9)$$

The greatest interest in the $u_i^{(k)}$ is in the $u_i^{(N-2)}$, because the minimum of the functional (2.2). The $u_i^{(N-2)}$ should be calculated after forming arrays of $\gamma^{(k)}$ and $\beta^{(k)}$ ($k = 1, 2, ..., N - 2$) by using the formulas

$$\begin{aligned}
&u_N^{(N-2)} = \gamma_N + \beta_N s_N, \; \gamma_N = \gamma^{(N-2)}, \; \beta_N = \beta^{(N-2)}, \\
&u_i^{(N-2)} = \gamma_i + \beta_i s_i, \; \gamma_i = \gamma_{i+1} + \gamma^{(i-2)}, \; \beta_i = \beta_{i+1} + \beta^{(i-1)}, \\
&i = N - 1, ..., 3, \\
&u_2^{(N-2)} = \gamma_3 + \beta_3 s_2.
\end{aligned} \qquad (3.10)$$

The number of operations in calculating $\gamma^{(k)}$ and $\beta^{(k)}$ ($k = 1, 2, ..., N - 2$) from Eqs. (3.5) and (3.6) and in calculating $u_i^{(N-2)}$ ($i = N - 1, ..., 2$) from Eqs. (3.10) on one node of the grid is independent of the number of grid nodes.

In a calculation without the limitation (2.8), the resultant $u_i^{(N-2)}$ are used as $\bar{u}_i$, the quantities $\tilde{\varphi}_i$ are replaced by $\tilde{\varphi}_i + \bar{u}_i$, and then the subsequent iterations are calculated.

If the limitation (2.8) is introduced, it can be formulated in the form of conditions on $\gamma^{(k)}$ and $\beta^{(k)}$, for example.

$$\max_{2 \leqslant i \leqslant k+2} |\dot{u}_i^{(k)}| = \max(|\gamma^{(k)} + \beta^{(k)} s_2|, \; |\gamma^{(k)} + \beta^{(k)} s_{k+2}|) \leqslant \epsilon, \; \epsilon = A/N. \qquad (3.11)$$

In this case, the process of forming the arrays of $\gamma^{(k)}$ and $\beta^{(k)}$ ($k = 1, 2, ..., N - 2$) stops when the conditions (3.11) are no longer fulfilled. Then the $u_i^{(k-1)}$ are calculated from Eqs. (3.10), where $N - 1$ is replaced by that value of k at which the conditions (3.11) are no longer fulfilled. The resultant $u_i^{(k-1)}$ are used for the $\bar{u}_i$, and the quantities $\tilde{\varphi}_i$ are replaced by $\tilde{\varphi}_i + \bar{u}_i$; then the subsequent iterations are calculated.

In the above iteration variant, the region where the variances of Eqs. (2.4) are self-equilibrating gradually expands along increasing node numbers of the grid. An analogous iteration variant can be formulated in which the region where the variances of Eqs. (2.4) are self-equilibrating gradually expands along decreasing node numbers.

Numerical experiments show that alternation of these iteration variants can substantially accelerate the convergence of the approximations.

4. APPROXIMATION ALGORITHM IF A BENDING MOMENT IS SPECIFIED AT THE END OF THE ROD

Problems in which a bending moment is specified at the end $s = l$ differ from those with a rotation angle specified at $s = l$ in Sec. 3 in that the condition $u_{N+1} = 0$ is replaced by the equation

$$d_N u_{N+1} + d_{N+1} u_{N+1} + f_{N+1} = 0. \qquad (4.1)$$

In this case convergence of the approximations can also be accelerated by alternating between two iteration variants.

In the variant where the region of self-equilibrating variances expands along increasing node numbers, $\bar{u}_i$ ($i = 1, 2, ..., N$) is calculated as in Sec. 3, and $\bar{u}_{N+1}$ is calculated from Eq. (4.1).

In the variant where the region of self-equilibrating variances expands along decreasing node numbers in the case of Eq. (4.1), the approximate solutions to Eqs. (2.4) are constructed in the form
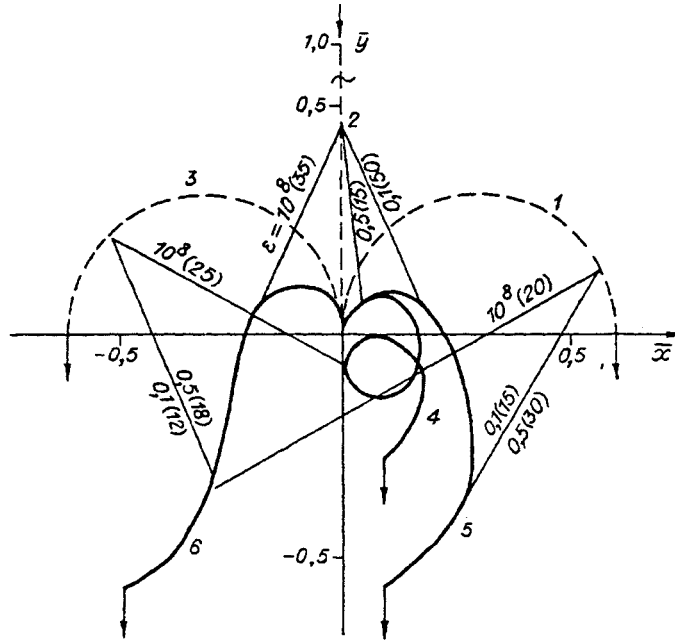
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Fig. 2

$$u^{(0)}_{N+1} = -f_{N+1}d^{-1}_{N+1}, \; u^{(0)}_i = 0, \; i = N, N-1, ..., 1,$$

$$u^{(k)}_{N+1} = u^{(k-1)}_{N+1} + \dot{u}^{(k)}_{N+1}, \; \dot{u}^{(k)}_{N+1} = -\dot{u}^{(k)}_N d_N d^{-1}_{N+1},$$

$$u^{(k)}_i = u^{(k-1)}_i + \dot{u}^{(k)}_i, \; \dot{u}^{(k)}_i = \gamma^{(k)}, \; i = N, ..., N+1-k, \tag{4.2}$$

$$u^{(k)}_i = \dot{u}^{(k)}_i = 0, \; i = N-k, N-k-1, ..., 1, \; k = 1, 2, ..., N-1,$$

where $\gamma^{(k)}$ is determined sequentially for $k = 1, 2, ..., N-1$ by the extremum condition on the functional (2.2) after u_i in it is replaced by $u^{(k)}_i$.

It is not difficult to show that the sequence (4.2) of approximate solutions to Eqs. (2.4) corresponds to a sequence of variances of these equations with a zero sum in a gradually expanding region. The quadratic part of the functional with respect to $u^{(k)}_i$ can be written as

$$\Delta = \frac{1}{2h} [D_{N+1/2}(\dot{u}^{(k)}_{N+1} - \dot{u}^{(k)}_N)^2 + D_{N-k+1/2}(\dot{u}^{(k)}_N)^2 + \frac{h^2}{4} \sum_{i=N-k}^{N} \tilde{T}_{i+1/2}(\dot{u}^{(k)}_i + \dot{u}^{(k)}_{i+1})^2].$$

Under the condition (3.8), it is obvious that

$$\Delta \geq \frac{1}{2h} \{ [2D_* - T_* h(l - \frac{3}{2}h)](\dot{u}^{(k)}_N)^2 - 2(D_* + \frac{1}{4}T_* h^2)\dot{u}^{(k)}_N \dot{u}^{(k)}_{N+1} + (D_* - \frac{1}{4}T_* h^2)(\dot{u}^{(k)}_{N+1})^2 \} \geq 0,$$

which can be equal to zero only if $\dot{u}_N^{(k)} = \dot{u}_{N+1}^{(k)} = 0$. Then, under the condition (3.8) and after u_i is replaced by $u_i^{(k)}$ in the functional (2.2), the functional will become a quadratic polynomial in $\gamma^{(k)}$ with a positive coefficient of $(\gamma^{(k)})^2$. Therefore we calculate the $\gamma^{(k)}$, and the sequence (4.2) corresponds to monotonically decaying values of the functional (2.2).

The values of $\gamma^{(k)}$ are calculated from recurrence formulas

$$E^{(1)}\gamma^{(1)} = f_N + c_N f_{N+1}d^{-1}_{N+1}, \; E^{(1)} = 2b_N - c_N d_N d^{-1}_{N+1},$$

$$E^{(k)}\gamma^{(k)} = f_{N-k+1} - c_{N-k+1}\gamma^{(k-1)}, \; E^{(k)} = E^{(k-1)} + a_{N-k+2} + 2b_{N-k+1} + c_{N-k+1},$$

$$k = 2, 3, ..., N-1.$$

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If the limitation (2.8) is not introduced, the array of the $\gamma^{(k)}$ (k = 1, 2, ... N − 1) are formed, and then the $u_i^{(N-1)}$ are calculated from the formulas

$$u_1^{(N-1)} = 0, \; u_i^{(N-1)} = u_{i-1}^{(N-1)} + \gamma^{(N+1-i)}, \; i = 2, ..., N,$$
$$u_{N+1}^{(N-1)} = -(f_{N+1} + d_N u_N^{(N-1)}) d_{N+1}^{-1}.$$

The resultant $u_i^{(N-1)}$ are used for the $\tilde{u}_i$, and the quantities $\tilde{\varphi}_i$ are replaced by $\tilde{\varphi}_i + \tilde{u}_i$; then the subsequent iterations are calculated.

In this case the limitation (2.8) can be formulated in the form

$$|\gamma^{(k)}| \leq \varepsilon, \; \varepsilon = A/N. \tag{4.3}$$

In a calculation with the limitation (4.3), the $u_i^{(k-1)}$ are used for the $\tilde{u}_i$, where k is the number at which the condition (4.3) is no longer fulfilled. The values of the $u_i^{(k-1)}$ are calculated from the formulas

$$u_i^{(k-1)} = 0, \; k = 1, 2, ..., N + 1 - k,$$
$$u_i^{(k-1)} = u_{i-1}^{(k-1)} + \gamma^{(N+1-i)}, \; i = N + 2 - k, ..., N,$$
$$u_{N+1}^{(k-1)} = -(f_{N+1} + d_N u_N^{(k-1)}) d_{N+1}^{-1}.$$

5. RESULTS OF NUMERICAL EXPERIMENTS

Numerical experiments were conducted in which the above algorithms were used to calculate equilibrium shapes. Here we examined various undeformed rod shapes, various initial approximations, various numbers of elements N, and various values of ε in the limitations (3.11) and (4.3). The calculations were done in terms of the dimensionless quantities

$$\bar{s} = s/l, \; \bar{F} = Fl^2/D, \; \bar{H} = Hl^2/D, \; \bar{p} = pl^3/D,$$
$$\bar{\Psi} = \Psi l/D, \; \bar{x} = x/l, \; \bar{y} = y/l.$$

Several typical results are shown in Fig. 2 and Table 1.

Figure 2 shows equilibrium shapes of an elastic rod (curves 4-6) acted on by a vertical force $\bar{H} = -40$ at the end of the rod $\bar{s} = 1$; the shape of the rod in the undeformed state is given by $\varphi_0 = \pi/2 - 3\pi\bar{s}$. The end of the rod $\bar{s} = 0$ is restrained. The bending moment is zero at $\bar{s} = 1$. Many different numerical experiments established that there are evidently no other equilibrium shapes of this rod with $\bar{H} = -40$.

The dashed curves 1-3 in Fig. 2 correspond to rod shapes described by $\varphi = \pi/2 - \pi\bar{s}$, $\varphi = \pi/2$, and $\varphi = \pi/2 + \pi\bar{s}$, which were used as initial approximations. The arrows show the transformation of the initial approximations into the corresponding equilibrium shapes in calculations with N = 120 and $\varepsilon = 10^8$, 0.5, and 0.1; the numbers in parentheses are the number of iterations after which the fourth digit after the decimal for the values of the functional (1.7) and the coordinates of the rod end cease to change. After 7-8 iterations, the errors in the energy functional and in the coordinate of the rod end were on the order of 10%.

In calculations with $\varepsilon = 0.1$, the functional (1.7) decreased monotonically for all iterations starting with the first; those with $\varepsilon = 10^8$ and 0.5 decreased monotonically only after several iterations.

Table 1 shows values of the functional (1.7) and (in parentheses) the coordinate of the rod end $\bar{s} = 1$ for N = 40, 80, and 120. These values almost coincide. The number of iterations required to obtain the equilibrium shapes is also almost independent of N.

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